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On the Generalization of the Modified Schnirelmann Density

A. R. FREEDMAN

Department of Mathematics, Simon Fraser University, Burnaby 2, B. C., Canada

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R. Stalley's modified Schnirelmann density is generalized to the setting of δ semigroups and k classes. If α^*, β^* and γ^* denote this density for sets A, B, and $C = A + B$ ($O \in A \cap B$), respectively then it is proved that (1) $\alpha^* + \beta^* > 1 \Rightarrow \gamma^* = 1$, (2) $\alpha^* + \beta^* \leqslant 1 \Rightarrow \gamma^* \geqslant \alpha_1 + \beta^*$ (where α_1 is the modified Erdös (or Kvarda) density of A, and, finally, (3) $\alpha^* + \beta^* \leqslant 1 \Rightarrow \gamma^* \geqslant [f(A)/f(A) + 1)]\, \alpha^* + \beta^*$, [where $f(A) = \min\{A(F) \mid F \in \mathscr{F}, A(F) < S(F)\}$].

1. INTRODUCTION

In this paper we introduce a generalization of the "modified Schnirelmann density" given by Stalley in [4]. We generalize this notion to the setting of δ semigroups and k classes which has been introduced and developed by the author in [1]. A familiarity with this paper is assumed below. The main interest in this modified density seems to be the unusual similarity between some of its results and some of those of asymptotic density.

Let S be a δ semigroups and $\mathscr{F}$ a k class on S. The following densities are defined for a subset A of S:

(i) The k density of A:

$$\alpha = d(A) = \operatorname{glb}\left\{\frac{A(F)}{A(F)} \,\middle|\, F \in \mathscr{F}\right\}$$

(ii) The Erdös density of A:

$$\alpha_1 = \operatorname{glb}\left\{\frac{A(F)}{S(F) + 1} \,\middle|\, F \in \mathscr{F}, A(F) < S(F)\right\}$$

for $A \neq S$ and $\alpha_1 = 1$ for $A = S$.

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(iii) For the case $S = I^n$ and $\mathscr{F} = \mathscr{K}(I^n)$, the asymptotic density of A:

$$\delta(A) = \lim_{N\to\infty} d[A \cup J(N)],$$

where $J(N) = \{(x_1, ..., x_n) \in I^n \mid \min\{x_1, ..., x_n\} \leqslant N\}$.

The density in (ii) has been studied only in the special case $S = I^n$, $\mathscr{F} = \mathscr{K}(I^n)$ by Kvarda in [3]. It is an open question regarding the characterization of all "density spaces" $(S, \mathscr{F})$ for which the following result of Kvarda holds:

LEMMA 1. (Kvarda). *If A, $B \subset I^n$, $O \in A \cap B$, $F \in \mathscr{K}(I^n)$,*

$$(A + B)(F) < S(F) \quad \textit{and} \quad F^* \subset \overline{A + B},$$

then

$$(A + B)(F) \geqslant \alpha_1[S(F) + 1] + B(F).$$

The condition $F^* \subset \overline{A + B}$ is a little stronger than what is actually needed, but it will suit our purposes.

The asymptotic density in (iii) has been introduced by the author in [2]. In particular the following two results are obtained:

(A) If $A, B \subset I^n, O \in A \cap B$ and $\delta(A) + \delta(B) > 1$, then $\delta(A + B) = 1$.

(B) If A, $B \subset I^n$, $O \in A \cap B$ and $\delta(A) + \delta(B) \leqslant 1$, then

$$\delta(A + B) \geqslant \alpha_1 + \delta(B).$$

We now define the modified k density of A to be

$$\alpha^* = \text{glb}\left\{\frac{A(F)}{S(F)} \;\middle|\; F \in \mathscr{F}, F^* \cap A \neq 0\right\},$$

provided A is infinite and $\alpha^* = 0$ otherwise. In the case $S = I, \mathscr{F} = \mathscr{K}(I)$ this clearly reduces to Stalley's definition: $\alpha^* = \text{glb}\{A(n)/n \mid n \in A\}$.

We show first that this density has a useful equivalent form which has gone hitherto unmentioned. For all $A \subseteq S$

$$\alpha^* = \text{glb}\left\{\frac{A(F) + 1}{S(F) + 1} \;\middle|\; F \in \mathscr{F}\right\}. \tag{1}$$

In order to prove (1) we shall require the following structure lemma which will be useful elsewhere as well:

LEMMA 2. *Let $A \subset S$, $F \in \mathscr{F}$ and suppose $A \backslash F \neq \varnothing$ (satisfied if A is infinite), then there exists a $G \in \mathscr{F}$ such that*

$$G^* \cap A \neq \varnothing, F \subset G \qquad \text{and} \qquad A(G) = A(F) + 1.$$

Moreover, if F is such that $F^* \subset \bar{A}$, then G may be chosen such that $S(G^* \cap A) = 1$ and $[G\backslash(G^* \cap A)]^* \subset \bar{A}$.

Proof. Let $a \in \mathrm{Min}(A\backslash F)$. Then we may take G to be $F \cup H(a)$. Now $a \in G^*$ since $G\backslash a = F \cup [H(a)\backslash a] \in \mathscr{F}$ and by the minimality of a it is clear that a is the only point in $A \cap (G\backslash F)$. Hence, all the required properties for G are proved except $[G\backslash(G^* \cap A)]^* \subset \bar{A}$ (if $F^* \subset \bar{A}$). By the results of Section 4, parts (ii), (iii) of [1] we have

$$(G\backslash a)^* \subset F^* \cup \{[H(a)\backslash a]\backslash F\} \subset \bar{A}.$$

But $\{a\} = G^* \cap A$ and so the lemma is proved.

Proof of (1). Let μ denote the r.h.s. of (1). For any $F \in \mathscr{F}$ we have

$$\frac{A(F)+1}{S(F)+1} = \frac{A(G)}{S(F)+1} \geqslant \frac{A(G)}{S(G)} \geqslant \alpha^*,$$

where G is given by Lemma 2. Hence, $\mu \geqslant \alpha^*$.

For the reverse inequality let $F \in \mathscr{F}$ such that $F^* \cap A \neq \varnothing$, and let $a \in F^* \cap A$. Then

$$\frac{A(F)}{S(F)} = \frac{A(F\backslash a)+1}{S(F\backslash a)+1} \geqslant \mu.$$

Another equivalent definition of α^* is given by:

$$\alpha^* = \mathrm{glb}\left\{\frac{A(F)}{S(F)}\,\middle|\, F \in \mathscr{F}_A\right\}, \tag{2}$$

where $\mathscr{F}_A = \{F \mid F \in \mathscr{F},\ S(F^* \cap A) = 1,\ [F\backslash(F^* \cap A)]^* \subset \bar{A}\}$. This is valid for infinite A, $A \neq S$.

Proof of (2). Let μ be the r.h.s. of (2). Clearly $\mu \geqslant \alpha^*$. Let $F \in \mathscr{F}$ such that $F^* \cap A \neq \varnothing$, and let $F_1 = \cup\{H(x) \mid x \in F \cap \bar{A}\}$. Then $A \supset F\backslash F_1 \neq \varnothing$. Applying Lemma 2 to F_1 we obtain a $G \in \mathscr{F}_A$ and we have

$$\frac{A(F)}{S(F)} = \frac{A(F_1)+S(F\backslash F_1)}{S(F_1)+S(F\backslash F_1)} \geqslant \frac{A(F_1)+1}{S(F_1)+1} \geqslant \frac{A(G)}{S(G)} \geqslant \mu.$$

Hence, $\alpha^* \geqslant \mu$ and (2) is proved.

For $F \in \mathscr{F}_A$ we define $F_0 = F\backslash(F^* \cap A)$. Then, for infinite $A \neq S$, we have the following equivalent form for α_1 :

$$\alpha_1 = \mathrm{glb}\left\{\frac{A(F_0)}{S(F_0)+1}\,\middle|\, F \in \mathscr{F}_A\right\}. \tag{3}$$

Proof of (3). Let μ denote the r.h.s. of (3). For $F \in \mathscr{F}_A$ we have $A(F_0) < S(F_0)$ and so $\mu \geqslant \alpha_1$. Now let $F \in \mathscr{F}$ such that $A(F) < S(F)$. Define F_1 and G as in the proof of (2). Then

$$\frac{A(F)}{S(F)+1} \geqslant \frac{A(F) - S(F \backslash F_1)}{S(F)+1-S(F \backslash F_1)} = \frac{A(F_1)}{S(F_1)+1}$$
$$= \frac{A(G)-1}{S(F_1)+1} \geqslant \frac{A(G_0)}{S(G_0)+1} \geqslant \mu.$$

The second to last inequality follows since $A(G) - 1 = A(G_0)$ and $S(F_1) \leqslant S(G) - 1 = S(G_0)$.

To conclude this section we note that

$$\alpha_1 \leqslant \alpha \leqslant \alpha^* \leqslant \delta(A).$$

The first two inequalities follow immediately from the definitions. The last inequality [where we are assuming that $S = I^n$ and $\mathscr{F} = \mathscr{K}(I^n)$] can be proved as follows: By Theorem 2.7 of [2] there exists a sequence (F_n) in $\mathscr{S}$ such that $\delta(A) = \lim_{n\to\infty} A(F_n)/S(F_n)$. Since $S(F_n) \to \infty$ we have $\lim_{n\to\infty} A(F_n)/S(F_n) = \lim_{n\to\infty} [A(F_n) + 1]/[S(F_n) + 1]$ and this last is $\geqslant \alpha^*$ by (1).

2. Some Density Inequalities

In this section we prove some inequalities involving the modified k density. We let A, B be subsets of S and will always assume that $O \in A \cap B$. Then the sumset $C = A + B$ is the set of all $a + b$ where $a \in A$, $b \in B$. Let α^*, β^* and γ^* denote the modified k density of A, B, and C, respectively.

We note that it follows immediately from (1) that, if $A \subset B$, then $\alpha^* \leqslant \beta^*$.

We proceed to prove the analogs of (A) and (B) above for modified k density.

THEOREM 1. *If* $\alpha^* + \beta^* > 1$, *then* $\gamma^* = 1$ (i.e. $C = S$).

Proof. Assume there is an $x \in \bar{C}$. We know $L(x) \in \mathscr{F}$, and by Theorem 2 of [1] we have

$$S[L(x)] - 1 \geqslant A[L(x)] + B[L(x)].$$

Applying Lemma 2 to $L(x)$ we obtain a G_1 such that $L(x) \subset G_1$,

$A(G_1) = A[L(x)] + 1$, $G_1^* \cap A \neq \varnothing$ and a G_2 such that $L(x) \subset G_2$, $B(G_2) = B[L(x)] + 1$, $G_2^* \cap B \neq \varnothing$. Hence,

$$S[L(x)] - 1 \geqslant A(G_1) + B(G_2) - 2$$

or

$$\begin{aligned} S[L(x)] + 1 &\geqslant A(G_1) + B(G_2) \\ &\geqslant \alpha^* S(G_1) + \beta^* S(G_2) \\ &\geqslant (\alpha^* + \beta^*)\{S[L(x)] + 1\}, \end{aligned}$$

so that $1 \geqslant \alpha^* + \beta^*$.

THEOREM 2. *Assume that the result of Lemma* 1 *holds for the density space* $(S, \mathscr{F})$ *[so, in particular, this theorem is true for* $S = I^n$, $\mathscr{F} = \mathscr{K}(I^n)$*]. If* $\alpha^* + \beta^* \leqslant 1$, *then* $\gamma^* \geqslant \alpha_1 + \beta^*$.

Proof. If $C = S$, then $\gamma^* = 1 \geqslant \alpha^* + \beta^* \geqslant \alpha_1 + \beta^*$. Suppose $C \neq S$ and let $F \in \mathscr{F}_C$. Since $F_0^* \subset \bar{C}$, we may apply Lemma 1 to obtain

$$\begin{aligned} C(F) = C(F_0) + 1 &\geqslant \alpha_1[S(F_0) + 1] + B(F_0) + 1 \\ &= \alpha_1 S(F) + B(F_0) + 1. \end{aligned} \tag{†}$$

Applying Lemma 2 to F_0 (for B) we get a $G \in \mathscr{F}$ such that

$$B(F_0) + 1 = B(G), \qquad F_0 \subset G, \quad G^* \cap B \neq \varnothing.$$

Also note that $S(F) \leqslant S(G)$. Combining this information with (†) we get

$$\begin{aligned} C(F) &\geqslant \alpha_1 S(F) + B(G) \\ &\geqslant \alpha_1 S(F) + \beta^* S(G) \\ &\geqslant (\alpha_1 + \beta^*)\, S(F), \end{aligned}$$

so that

$$C(F)/S(F) \geqslant \alpha_1 + \beta^*$$

for each $F \in \mathscr{F}_C$. By (2) the theorem follows.

We can go a little further with the modified k density but no analogous result for the asymptotic density has yet been proved (except, of course, in the 1-dimensional case). We are refering here to our generalization of Stalley's Theorem 2 in [4].

For $A \not\subseteq S$ let $f(A) = \min\{A(F) \mid F \in \mathscr{F},\ A(F) < S(F)\}$. For $S = I$, $\mathscr{F} = \mathscr{K}(I)$ this clearly reduces to k if $0, 1, \ldots, k \in A$ and $k + 1 \notin A$.

LEMMA 3. $\alpha_1 \geqslant [f(A)/(f(A) + 1)]\, \alpha^*$.

Proof. Using (3) and (2) above we have

$$\begin{aligned}
\alpha_1 &= \operatorname{glb}\left\{\frac{A(F_0)}{S(F_0)+1}\,\middle|\, F \in \mathscr{F}_A\right\} \\
&= \operatorname{glb}\left\{\frac{A(F_0)}{A(F)}\cdot\frac{A(F)}{S(F_0)+1}\,\middle|\, F \in \mathscr{F}_A\right\} \\
&= \operatorname{glb}\left\{\frac{A(F_0)}{A(F_0)+1}\cdot\frac{A(F)}{S(F)}\,\middle|\, F \in \mathscr{F}_A\right\} \\
&\geqslant \operatorname{glb}\left\{\frac{A(F_0)}{A(F_0)+1}\,\middle|\, F \in \mathscr{F}_A\right\}\cdot \operatorname{glb}\left\{\frac{A(F)}{S(F)}\,\middle|\, F \in \mathscr{F}_A\right\} \\
&\geqslant \frac{f(A)}{f(A)+1}\cdot\alpha^*.
\end{aligned}$$

Combining Theorem 2 and Lemma 3 we obtain

THEOREM 3. *Under the assumption of Theorem 2 we have*

$$\gamma^* \geqslant [f(A)/(f(A)+1)]\,\alpha^* + \beta^*.$$

3. ANOTHER DENSITY

We conclude this paper by reporting on one of the densities which generalizes the modified Schnirelmann density, however, not very successfully. Nevertheless, there seems to be some challenging aspects of this particular density.

We confine ourselves to the case $S = I^n$ and $\mathscr{F} = \mathscr{K}(I^n)$. For a set $A \subseteq I^n$ such that the k density of $A \cup \operatorname{Min}(I^n \backslash O)$ is nonzero we define

$$\alpha^0 = \operatorname{glb}\left\{\frac{A(F)}{S(F)}\,\middle|\, F \in \mathscr{K}, F^* \subset A\right\}, \tag{4}$$

and $\alpha^0 = 0$ otherwise. [The somewhat complicated condition required in order to apply the formula (4) becomes evident when one studies the "Question" asked below.] Thus, we have merely required that all of the corner points of F be in A rather than at least one corner point as in the definition of α^*. In the 1-dimensional case, however, it is all the same so that this new density also generalizes the modified Schnirelmann density.

The density given in (4) has the rather unpleasant property that, for $n > 1$, $A \subseteq B$ need not imply that $\alpha^0 \leqslant \beta^0$. An example of this is given in I^2 by

$$A = \{(x, y) \mid x \neq 1 \quad \text{or} \quad (x, y) = (1, 0)\}, \qquad B = A \cup \{(1, 100)\}.$$

We easily calculate that $\alpha^0 = 2/3$ while $\beta^0 = 102/201 \sim 1/2$.

The author has not been able to answer the following basic question:

QUESTION. *What is the greatest lower bound b of numbers x such that, if $O \in A \cap B$ and $\alpha^0 + \beta^0 > x$, then $A + B = I^n$?*

For $n = 1$, the answer is, of course, $b = 1$. For $n > 1$, the following example shows that $b \geqslant 4/3$. (The example is for I^2 but can easily be extended to any I^n, $n > 1$): Let

$$A = B = \{(x, y) | \; x \neq 1 \quad \text{or} \quad (x, y) = (1, 0)\} \backslash (0, k) \quad (k > 1).$$

Then $\alpha^0 = \beta^0 = (2k + 1)/(3k + 2)$ and $\alpha^0 + \beta^0 = (4k + 2)/(3k + 2)$. But $C \neq I^2$ as $(1, k) \notin C$.

We can prove the following partial result which lends support to the conjecture that $b = 4/3$.

THEOREM 4. *Let $A, B \subseteq I^n (n > 1)$, such that $O \in A \cap B$ and $C = A + B \neq I^n$. Suppose there exists $g = (g_1, g_2, \ldots, g_n) \in \bar{C}$ such that, for some i, $g + e_i \in A \cap B$ and $g_i \geqslant g_j$ $(j = 1, 2, \ldots, n)$ where $e_i = (\delta_i, \delta_{2i}, \ldots, \delta_{ni})$. Then*

$$\alpha^0 + \beta^0 \leqslant 1 + \frac{2^{n-1} - 1}{3.2^{n-1} - 1} \left(<\tfrac{4}{3}\right),$$

and this is best possible.

Proof. Without loss of generality we may assume $g' = (g_1 + 1, g_2, \ldots, g_n) \in A \cap B$ and that $g_1 \geqslant g_i (i = 1, 2, \ldots, n)$. Now, using Theorem 2 of [1], we obtain

$$\begin{aligned} A[L(g')] + B[L(g')] &\leqslant A[L(g)] + B[L(g)] + 2S[L(g') \backslash L(g)] \\ &= A[L(g)] + B[L(g)] + 2 \prod_2^n (g_i + 1) \\ &\leqslant S[L(g)] - 1 + 2 \prod_2^n (g_i + 1) \\ &= S[L(g')] + \prod_2^n (g_i + 1) - 1. \end{aligned}$$

Upon dividing by $S[L(g')]$ we get

$$\begin{aligned} \alpha^0 + \beta^0 &\leqslant \frac{A[L(g')]}{S[L(g')]} + \frac{B[L(g')]}{S[L(g')]} \\ &\leqslant 1 + \left[\prod_2^n (g_i + 1) - 1\right] \Big/ \left[(g_1 + 2) \prod_2^n (g_i + 1) - 1\right]. \end{aligned}$$

We have to maximize the fraction in the expression on the right hand side. Let

$$E = \frac{X - 1}{(g_1 + 2)X - 1},$$

where $X = \prod_2^n (g_i + 1)$. Clearly $E < 1/(g_1 + 2)$; and so for any $\xi > 0$, we have

$$\frac{X - 1 + \xi}{(g_1 + 2)X - 1 + (g_1 + 2)\xi} > E.$$

Now suppose that $g_i > g_j$ for some $i, j \geqslant 2$. Then, if we increase g_j by 1, the new E we get is

$$\left[X + \frac{X}{g_j + 1} - 1\right] \Big/ \left[(g_1 + 2)X + \left(\frac{g_1 + 2}{g_j + 1}\right)X - 1\right] > E.$$

Hence, in order to maximize E, we may assume that $g_2 = g_3 = \cdots = g_n$ and thus, since $g_1 \geqslant g_i$, that $g_1 = g_2 = \cdots = g_n$. It follows that the maximum value for E exists and occurs among the numbers

$$(k^{n-1} - 1)/((k + 1)\, k^{n-1} - 1)$$

for integers $k \geqslant 1$. This can easily be seen to be greatest for $k = 2$. Therefore we have

$$\alpha^0 + \beta^0 \leqslant 1 + (2^{n-1} - 1)/(3 \cdot 2^{n-1} - 1).$$

The result is shown to be best possible by the following examples:

$$A = I^n \backslash (1, 1, \ldots, 1), \qquad B = I^n \backslash L(1, 1, \ldots, 1).$$

These sets satisfy the hypothesis of the theorem with $g = (1, 1, \ldots, 1)$. We calculate that

$$\alpha^0 = (3 \cdot 2^{n-1} - 2)/(3 \cdot 2^{n-1} - 1)$$

and

$$\beta^0 = 2^{n-1}/(3 \cdot 2^{n-1} - 1),$$

so that

$$\alpha^0 + \beta^0 = 1 + (2^{n-1} - 1)/(3 \cdot 2^{n-1} - 1).$$

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